REMARK ON A THEOREM OF AHARONOV AND WALSH

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ABSTRACT

The following theorem is proved: there is a function $f(z)$ analytic in $|z| < 1$ and having the natural boundary $|z| = 1$ such that for an infinite sequence of rational functions of degree n, $r_n(z) = p_n(z)/q_n(z)$, the inequality

$$(*) \qquad |f(z) - r_n(z)| < \varepsilon_n$$

holds in the closed unit circle $|z| \leqq 1$. Here $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ is any sequence of positive numbers, tending to zero as n approaches infinity. This theorem is a refinement of a theorem of Aharonov and Walsh, who showed the existence of an $f(z)$ satisfying $(*)$ in $|z| \leqq 1$ (with an infinite sequence $\{r_n(z)\}$) but having the natural boundary $|z| = 3$.

Let $f(z) = a_0 + a_1 z + \cdots$ be a function regular for $|z| < 1$; further, put $S_n(z) = a_0 + a_1 z + \cdots + a_n z^n$. It is a consequence of the Cauchy-Hadamard formula that if

$$(1) \qquad \lim_{n \to \infty} \max_{z \in D} |f(z) - S_n(z)|^{1/n} = 0$$

in any domain D contained in the circle $|z| < 1$, then $f(z)$ is an entire function.

In a recent paper [1], D. Aharonov and J. L. Walsh considered the question whether the analogous statement holds if the powers $1, z, z^2, \cdots$ are replaced by more general functions regular for $|z| < 1$. They solved this question in the negative sense for

$$(2) \qquad g(z) = \sum_{n=1}^{\infty} \frac{A_n}{z - z_n},$$

where $|z_n| > 1$. (If the A_n's tend to zero rapidly enough, then $g(z)$ defined by (2) is

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analytic in $|z| < 1$.) Among several similar examples they give an example for which $g(z)$ defined by (2) has the natural boundary $|z| = 3$ and still

$$\lim_{n\to\infty} \left| g(z) - \sum_{k=1}^{n} \frac{A_k}{z - z_k} \right|^{1/n} = 0 \tag{3}$$

uniformly in $|z| \leqq 1$.

Actually their construction gives, instead of (3), the stronger statement

$$\max_{|z|\leqq 1} \left| g(z) - \sum_{k=1}^{n} \frac{A_k}{z - z_k} \right| < \varepsilon_n, \tag{3'}$$

where $\varepsilon_1, \varepsilon_2, \cdots$ is an arbitrary sequence of positive numbers; on the other hand, it seems to be essential that (3′) holds for $|z| \leqq 1$ and the natural boundary of $g(z)$ is $|z| = 3$, or by a slight modification of the construction, any circle $|z| = \rho$ with $\rho > 1$.

In the present short note I give an example of a function $g(z)$,for which (3′) holds and $|z| = 1$ is the natural boundary of $g(z)$.

Construction of the $g(z)$

Let $\varepsilon_1, \varepsilon_2, \cdots$ be any sequence of positive numbers; $\delta_1, \delta_2, \cdots$ a sequence of positive numbers satisfying

$$\sum_{k=n+1}^{\infty} \delta_k \leqq \varepsilon_n. \tag{4}$$

Put $z_{kl} = (1 + 1/k)\exp(2\pi i l/2^k)$ $(k = 1,2,\cdots;\ l = 0,1,\cdots,2^k - 1)$; and

$$g(z) = \sum_{k=1}^{\infty} \frac{\delta_k}{k2^k} \sum_{l=0}^{2^k-1} \frac{1}{z_{kl} - z}. \tag{5}$$

I show that this $g(z)$ satisfies all our requirements. By (4), (5) and by the fact that for $|z| \leqq 1$ $|z_{kl} - z| \geqq 1/k$ holds, we have (3′).

It remains only to show that $g(z)$ has the natural boundary $z| = 1$. Obviously

$$\sum_{k=1}^{n} \frac{\delta_k}{k2^k} \sum_{l=0}^{2^k-1} \frac{1}{z_{kl} - z} = g_n(z)$$

is regular for $|z| \leqq 1$. Since for any fixed ζ of the form $\zeta = \exp 2\pi i\, l/2^n$,

$$\frac{g(\zeta z) - g_n(\zeta z)}{\zeta} = g(z) - g_n(z),$$

for any natural n and integer l $(0 \leqq l < 2^n)$, it suffices to show that $g(z)$, defined by

(5) cannot be continued beyond $z = 1$. Develop $g(z)$ into a power-series around the point $\frac{1}{2}$. Simple computation gives $g(z) = \sum_{n=0}^{\infty} \mathscr{C}_n (z - \frac{1}{2})^n$ where

$$\mathscr{C}_n = \sum_{k=1}^{\infty} \frac{1}{2^k k} \sum_{l=0}^{2^k - 1} \frac{1}{(z_{kl} - \frac{1}{2})^{n+1}} \tag{6}$$

One has

$$\begin{aligned} \sum_{l=0}^{2^k-1} \frac{1}{(z_{kl} - \frac{1}{2})^{n+1}} &= \sum_{m=0}^{\infty} \binom{n+1+m}{m} \frac{1}{2^m} \sum_{l=0}^{2^k-1} z_{kl}^{-(n+1-m)} \\ &= \sum_{\substack{m=0 \\ 2^k | n+1+m}}^{\infty} \binom{n+1+m}{m} \frac{1}{2^{m-k}} \frac{1}{(1+1/k)^{n+1+m}}. \end{aligned} \tag{7}$$

By (7) all $\mathscr{C}_n$'s are positive. On the other hand, for $n = 2^{k-1} - 1$ we have by (6) and (7)

$$\mathscr{C}_n > \frac{1}{n} \frac{1}{2^n} \binom{2n}{n} \frac{1}{(1+1/k)^{2n}},$$

or, by Stirling's formula,

$$\mathscr{C}_n > C \frac{1}{n^{\frac{3}{2}}} 2^n \frac{1}{(1+1/\log n)^{2n}}, \tag{8}$$

C being a numerical constant. (8) yields

$$\lim_{k \to \infty} \left| \mathscr{C}_{2^{k-1}-1} \right|^{1/(2^{k-1}-1)} = 2,$$

hence $z = 1$ is a singular point of $g(z)$.

Reference

1. D. Aharonov and I. L. Walsh, *Some examples in degree of approximation by rational functions*, Trans. Amer. Math. Soc. **139** (1971), 428–444.

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